ANALYTIC FUNCTIONS IN THREE DIMENSIONS*

BY

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1. **Introduction.** In this paper the authors discuss certain analogies in three-dimensional space of the classical theory of analytic functions in two dimensions.

It is possible that isolated instances of such analogies have occured to many mathematicians, but so far as we are aware no systematic investigation of this subject has ever been published.

The subject is approached by means of the *stretching factor* of a transformation at a point and the related generalized Tissot indicatrix.

In two dimensions it has been shown that the Tissot indicatrix for an analytic function is a circle, and the condition that the Tissot indicatrix be a circle leads immediately to the Cauchy-Riemann equations.†

The corresponding condition in three dimensions‡ leads to equations analogous to the Cauchy-Riemann equations, and the corresponding function or transformation is conformal. Conformal transformations of space have, of course, received considerable attention, but their analogies with analytic functions in two dimensions do not appear to have been sufficiently emphasized.

A satisfactory generalization of analytic functions by way of the derivative property seems difficult. A definition of multiplication or the equivalent is needed and the commutative property is desirable.§

The derivative property may, however, be investigated from other angles, and some of these may possibly be extensible to three dimensions. In a

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† See Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, ser. 9, vol. 2 (1923), pp. 327–342.

‡ We shall discuss in this paper in detail only the case of equal axes. The general case is mentioned and will be discussed in detail in another paper. Some, but not all, of these details are similar to those developed in the paper by Hedrick, Ingold, and Westfall just cited.

§ The following definition of multiplication was considered: Given $W = (u, v, w)$, $R = (p, q, r)$, $WR = [u, v, w]\,[p, q, r] = [pu + qw + rv,\ pv + qu + rw,\ ru + pw + qv]$. According to this definition, the reciprocal of $Z = (x, y, z)$ is (a, b, c) where $a = (x^2 - yz)/\Delta$, $b = (z^2 - xy)/\Delta$, $c = (y^2 - zx)/\Delta$, $\Delta = (x + y + z)\,(x^2 + y^2 + z^2 - yz - zx - xy)$.

With this definition it is possible to define the derivative of $W = f(z)$ and the conditions for the (unique) existence of the derivative are easily obtained. The peculiar nature of ordinary singularities (poles) as revealed by the forms for a, b, c above seems to make a theory based on this definition undesirable. A similar objection has been noticed by G. Y. Rainich (Bulletin of the American Mathematical Society, vol. 30 (1924), p. 8) to a generalization based on quaternionic multiplication.

future paper the authors hope to give some results of investigations along this line; but for the present we devote our attention to the more fundamental analogies which spring from the generalization of the Cauchy-Riemann equations.

2. **The expansion factor in three dimensions.** As in two dimensions, a function

$$(u, v, w) = F(x, y, z) \tag{1}$$

corresponds to a transformation from the xyz-space to the uvw-space.

If arc length in the xyz-space is given by the equation

$$ds^2 = dx^2 + dy^2 + dz^2$$

and in the uvw space by

$$d\sigma^2 = du^2 + dv^2 + dw^2,$$

the expansion factor of the function $R = d\sigma^2/ds^2$ is easily computed. We have

$$du = u_x\, dx + u_y\, dy + u_z\, dz, \tag{2}$$

with similar expressions for dv and dw, and consequently

$$d\sigma^2 = E_{11}\, dx^2 + E_{22}\, dy^2 + E_{33}\, dz^2 + 2E_{12}\, dx\, dy + 2E_{23}\, dy\, dz + 2E_{31}\, dz\, dx, \tag{3}$$

where

$$E_{11} = u_x^2 + v_x^2 + w_x^2, \qquad E_{12} = u_x u_y + v_x v_y + w_x w_y, \text{ etc.}$$

Thus we have

$$R = \frac{E_{11}\, dx^2 + \cdots}{dx^2 + dy^2 + dz^2}. \tag{4}$$

3. **Stationary values of R.** The expansion factor R will, in general, have different values in different directions. The equations for the extremes of R, when they exist, may be written in the form

$$\begin{aligned} E_{11}\, dx + E_{12}\, dy + E_{13}\, dz &= R\, dx, \\ E_{21}\, dx + E_{22}\, dy + E_{23}\, dz &= R\, dy, \\ E_{31}\, dx + E_{32}\, dy + E_{33}\, dz &= R\, dz. \end{aligned} \tag{5}$$

Eliminating dx, dy, dz we obtain a cubic for R,

$$\begin{vmatrix} E_{11} - R & E_{12} & E_{13} \\ E_{21} & E_{22} - R & E_{23} \\ E_{31} & E_{32} & E_{33} - R \end{vmatrix} = 0. \tag{6}$$

Let R and ϱ be two distinct solutions of this cubic and let dx, dy, dz and $\delta x, \delta y, \delta z$ be the two sets of differentials corresponding to them. Multiply the equations (5) in order by $\delta x, \delta y, \delta z$ and add; then if we reduce by equations (5) applied to ϱ and $\delta x, \delta y, \delta z$, we find

$$dx(\varrho\,\delta x) + dy(\varrho\,\delta y) + dz(\varrho\,\delta z) = R(dx\,\delta x + dy\,\delta y + dz\,\delta z),$$

or

$$(\varrho - R)(dx\,\delta x + dy\,\delta y + dz\,\delta z) = 0, \tag{7}$$

and since ϱ and R are distinct this shows that the two corresponding directions are orthogonal. When the solutions of the cubic in R are all distinct the corresponding directions determined by equations (5) are also distinct and mutually orthogonal. We shall call them the *principal directions* of the function.

A curve whose direction at each point coincides with a principal direction is called a *characteristic curve* of the function. When the solutions of the cubic in R are distinct there are three mutually orthogonal families of characteristic curves.

4. **Analytic functions.** The case in which R has the same value in all directions corresponds to the analytic case in two dimensions. For this case it is necessary and sufficient that

$$\begin{aligned} R &= \frac{d\sigma^2}{ds^2} \\ &= \frac{E_{11}\,dx^2 + E_{22}\,dy^2 + E_{33}\,dz^2 + 2(E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx)}{dx^2 + dy^2 + dz^2} \end{aligned} \tag{8}$$

be independent of the direction, i. e., of dx, dy, dz. We can now determine the conditions on the E_{ij} by giving special values to dx, dy, dz or their ratios. If $dx = dy = 0$, and $dz = 1$, we have $E_{33} = R$. Similarly $E_{11} = R$, $E_{22} = R$. Using these values we have

$$R = \frac{E(dx^2 + dy^2 + dz^2) + 2(E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx)}{dx^2 + dy^2 + dz^2}$$

where E denotes the common value of E_{11}, E_{22} and E_{33}.

It follows that $E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx$ must vanish for all values of dx, dy, dz and hence

$$E_{12} = E_{23} = E_{31} = 0.$$

Thus the necessary and sufficient conditions* that R have the same value in all directions are that

$$(9) \qquad E_{11} = E_{22} = E_{33} = E, \text{ and } E_{12} = E_{23} = E_{31} = 0.$$

It is well known that the corresponding transformation is conformal.†

It is also known that the solutions of the equation $d\sigma^2 = Eds^2$ may always be obtained by a succession of inversions.‡

5. **Generalization of the Cauchy-Riemann equations.** From the above we have immediately (for the conformal case)

$$(10) \qquad \begin{aligned} u_x^2 + v_x^2 + w_x^2 &= E, \\ u_y^2 + v_y^2 + w_y^2 &= E, \\ u_z^2 + v_z^2 + w_z^2 &= E; \end{aligned}$$

$$(11) \qquad \begin{aligned} u_x u_y + v_x v_y + w_x w_y &= 0, \\ u_y u_z + v_y v_z + w_y w_z &= 0, \\ u_z u_x + v_z v_x + w_z w_x &= 0. \end{aligned}$$

From (11) we have

$$(12) \qquad u_x = \lambda \begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix}, \qquad v_x = \lambda \begin{vmatrix} w_y & u_y \\ w_z & u_z \end{vmatrix}, \qquad w_x = \lambda \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix},$$

where λ is some multiplier, in general a function of x, y, z.

When these are substituted in the first of equations (10) it is found after reduction by means of the second and third of equations (10) that $\lambda = 1/\sqrt{E}$.

It is now easy to show that

$$(13) \quad J = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = u_x \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} + u_y \begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix} + u_z \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = E^{3/2},$$

* These conditions might be obtained by expressing the condition that the characteristic ellipsoid should reduce to a sphere.

† See Eisenhart, *Differential Geometry*, p. 99. The proof for three or more dimensions is wholly analogous to the proof for two dimensions.

‡ Sir William Thompson seems to have been the first to use the method of inversion to find functions u, v, w, satisfying the equation

$$du^2 + dv^2 + dw^2 = E(dx^2 + dy^2 + dz^2)$$

(see Liouville's Journal, vol. 10, p. 364). Later (Liouville's Journal, vol. 15, p. 103) Liouville announced that the only solutions were those given by Thompson's method. Professor Eisenhart has recently called our attention to Bianchi's generalization of Liouville's theorem. See *Lezioni di Geometria Differenziale*, 2d edition, vol. 1, pp. 375, 376.

and also that

$$(14)\qquad \begin{gathered} u_x^2+u_y^2+u_z^2 = v_x^2+v_y^2+v_z^2 = w_x^2+w_y^2+w_z^2 = E, \\ u_x v_x + u_y v_y + u_z v_z = v_x w_x + v_y w_y + v_z w_z = w_x u_x + w_y u_y + w_z u_z = 0. \end{gathered}$$

Equations (12) seem to be the natural generalizations of the Cauchy-Riemann equations. In complete form they are

$$(15)\qquad \begin{aligned} u_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix}, & v_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_y & u_y \\ w_z & u_z \end{vmatrix}, & w_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}, \\ u_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_z & w_z \\ v_x & w_x \end{vmatrix}, & v_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_z & u_z \\ w_x & u_x \end{vmatrix}, & w_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}, \\ u_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix}, & v_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_x & u_x \\ w_y & u_y \end{vmatrix}, & w_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}. \end{aligned}$$

6. Generalization of Laplace's equation. From the generalized Cauchy-Riemann equations we have

$$(16)\qquad \begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix} = \sqrt{E}\,u_x, \quad \begin{vmatrix} v_z & w_z \\ v_x & w_x \end{vmatrix} = \sqrt{E}\,u_y, \quad \begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix} = \sqrt{E}\,u_z,$$

where

$$E = u_x^2 + u_y^2 + u_z^2;$$

and from these we obtain immediately (by taking derivatives and adding) the equation

$$(17)\qquad \frac{\partial(\sqrt{E}\,u_x)}{\partial x} + \frac{\partial(\sqrt{E}\,u_y)}{\partial y} + \frac{\partial(\sqrt{E}\,u_z)}{\partial z} = 0.$$

The functions v and w must satisfy the same differential equation if $(u, v, w) = F(x, y, z)$ represents a conformal transformation.

This equation may justly be regarded as a generalization of the equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ for the two-dimensional case.

The close analogy between this equation and the traditional Laplace equation both in two and in three dimensions is at once evident. This one reduces to the customary form if E is a constant.

In future papers we expect to continue this study of the analogies between functions in three dimensions and functions in two dimensions. In particular we expect to obtain extensions of the well known Beltrami equations, and to give still other generalizations of Laplace's equation with applications to the problem of conjugate functions.

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